3.7 Part 1 Characteristics of Rational Functions

A rational function is the quotient of two polynomial functions.

The equation looks like $f(x) = \frac{p(x)}{q(x)}$.

Domain: Remember the denominator cannot equal zero.

For example: The domain for $f(x) = \frac{x+4}{x-7}$ is $(-\infty, 7) \cup (7, \infty)$.

Example 1: Find the domain of $g(x) = \frac{x^2+4x}{x^2-2}$.

$x^2 - 2 \neq 0$

$\sqrt{x^2} \neq \sqrt{2}$

$x \neq \pm\sqrt{2}$

$(-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty)$

Vertical Asymptotes & Holes

Vertical asymptotes are found by setting the denominator equal to zero UNLESS a factor occurs in both the numerator and denominator (this is where holes occur).

For example: The function $h(x) = \frac{x^2 + 5x - 6}{x^2 + 3x - 4}$ has a vertical asymptote at $x = 4$ and a hole at $(1, \frac{7}{5})$.

$h(x) = \frac{(x+6)(x-1)}{(x-1)(x+4)} = \frac{x+6}{x+4}$

VA

$x+4=0$

VA: $x=-4$

Hole @ $x=1$

$\frac{1+6}{1+4} = \frac{7}{5}$

Example 2: Find any vertical asymptote(s) and/or hole(s) of each.

a) $k(x) = \frac{4x}{2x^2 + x}$

$k(x) = \frac{4x}{x(2x+1)}$

$k(x) = \frac{4}{2x+1}$ VA

Hole @ $x=0$

$k(0) = \frac{4}{2(0)+1} = \frac{4}{1} = 4$

Hole $(0, 4)$

VA: $x = -\frac{1}{2}$

b) $y = \frac{5x^3}{2x^2 + 7x + 6}$

$y = \frac{5x^3}{(x+2)(2x+3)}$

No Holes

VA: $x=-2$

$x = -\frac{3}{2}$

c) $f(x) = \frac{3x^2 - 22x + 7}{x^2 - 3x - 28}$

$f(x) = \frac{(x-7)(3x-1)}{(x-7)(x+4)}$

$f(x) = \frac{3x-1}{x+4}$

Hole $(7, \frac{20}{11})$

$f(7) = \frac{3(7)-1}{7+4} = \frac{20}{11}$

VA: $x=-4$

Horizontal Asymptotes

To find the horizontal asymptotes, you must first find the degree of the numerator and denominator.

TOP HEAVY: degree of num. > degree of den. ⇒ No HA

BOTTOM HEAVY: degree of num. < degree of den. ⇒ HA is $y = 0$

SAME: degree of num. = degree of den. ⇒

HA is $y = \frac{a}{b}$ (a is top leading coefficient, b is bottom leading coefficient)

Example 3: Find the horizontal asymptote, if any, of each.

a) $k(x) = \dfrac{4x}{2x^2 + x}$

bottom heavy

HA: $y=0$

b) $y = \dfrac{5x^3}{2x^2 + 7x + 6}$

top heavy

no HA

c) $f(x) = \dfrac{3x^2 - 22x + 7}{1\,x^2 - 3x - 28}$

same

HA: $y=3$

Slant (or Oblique) Asymptotes

TOP HEAVY

If the degree of the numerator is greater than the degree of the denominator by one there is a slant asymptote.

To find the slant asymptote, you must divide using long division.

Example 4: Determine whether the graph will have a horizontal or slant asymptote. Then find it.

a) $f(x) = \dfrac{3x^2 + 1}{x^2 + x + 10}$

same

HA: $y=3$

b) $g(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

top heavy by 1

$$\begin{array}{r|l} & 2x - 1 \\ x-2 & 2x^2 - 5x + 5 \\ & -2x^2 + 4x \\ \hline & -x + 5 \\ & +x - 2 \\ \hline & 3 \end{array}$$

SA: $y=2x-1$

Example 4: Determine whether the graph will have a horizontal or slant asymptote. Then find it.

c) $y = \dfrac{3x^2 + 2x + 8}{x + 4}$

$$\begin{array}{r|l} & 3x - 10 \\ x+4 & 3x^2 + 2x + 8 \\ & -3x^2 - 12x \\ \hline & -10x + 8 \\ & +10x + 40 \\ \hline \end{array}$$

SA: $y=3x-10$

d) $w(x) = \dfrac{2x - 7}{4x^2 - x + 1}$

HA: $y=0$

Intercepts

To find an asymptote, set the other variable equal to zero & solve.

$f(x) = x - 5$

Example 5: Find the domain, all asymptotes, holes, & intercepts.

$$f(x) = \frac{x^3 - 2x^2 - 15x}{x^2 + 3x} = \frac{x(x^2-2x-15)}{x(x+3)} = \frac{\cancel{x}(x-5)\cancel{(x+3)}}{\cancel{x}\cancel{(x+3)}}$$

Domain: $x \neq 0, -3$ $(-\infty,-3) \cup (-3,0) \cup (0,\infty)$

Holes: $(0,-5)$, $(-3,-8)$

VA: none

HA: none

SA: $y = x - 5$

x-int: $(5, 0)$

y-int: none

x-int

$0 = x - 5$

$5 = x$

$$\begin{array}{r} x - 5 \\ x^2+3x \overline{) \, x^3 - 2x^2 - 15x + 0} \\ -(x^3 + 3x^2) \\ \hline -5x^2 - 15x \\ +5x^2 + 15x \\ \hline 0 \end{array}$$

y-int

$y = 0 - 5$

$y = -5$

already a hole

Example 6: Find the domain, all asymptotes, holes, & intercepts.

$$f(x) = \frac{5x^2 - 1}{x^2 - 16} = \frac{5x^2-1}{(x-4)(x+4)}$$

Domain: $x \neq \pm 4$ $(-\infty,-4) \cup (-4,4) \cup (4,\infty)$

No Holes

VA: $x = -4$, $x = 4$

HA: $y = 5$

SA: none

x-int: $\left(\frac{\sqrt{5}}{5}, 0\right)$ $\left(-\frac{\sqrt{5}}{5}, 0\right)$

y-int: $\left(0, \frac{1}{16}\right)$

x-int

$x^2-16 \cdot 0 = \frac{5x^2-1}{\cancel{x^2-16}} \cdot \cancel{x^2-16}$

$0 = 5x^2 - 1$

$1 = 5x^2$

$\sqrt{\frac{1}{5}} = \sqrt{x^2}$

$\pm\frac{1 \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = x$

$\pm\frac{\sqrt{5}}{5} = x$

y-int

$f(0) = \frac{5 \cdot 0^2 - 1}{0^2 - 16}$

$\frac{1}{16}$