3.6 Complex Zeros and the Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that every polynomial with complex coefficients has at least one complex zero. Since any real number is also a complex number, this theorem also works for polynomials with real coefficients.

The Complete Factorization Theorem states that every polynomial can be factored, although you may have to use complex numbers.

A complex number is a real number ± an imaginary number.

examples: $3 + 2i$ $-6i$ $-4 - i$

Example 1

Find all solutions of the equation by factoring.

$P(x) = x^4 + 4x^2$

$0 = x^2(x^2 + 4)$

$\sqrt{x^2} = \sqrt{0}$

$\boxed{x = 0}$

$x^2 + 4 = 0$

$-4 \quad -4$

$\sqrt{x^2} = \sqrt{-4}$

$\boxed{x = \pm 2i}$

Example 2

Find all solutions of the equation by factoring.

$x^6 - 64 = 0$

$(x^3 - 8)(x^3 + 8) = 0$

$(x)^3 \quad (2)^3 \quad (x)^3 \quad (2)^3$

$(a-b)(a^2+ab+b^2) \quad (a+b)(a^2-ab+b^2)$

$(x-2)(x^2+2x+4)(x+2)(x^2-2x+4) = 0$

$x-2=0$

$\boxed{x=2}$

$x^2+2x+4=0$

$x=\frac{-2\pm\sqrt{(2)^2-4(1)(4)}}{2(1)}$

$x=\frac{-2\pm\sqrt{-12}}{2}$

$x=\frac{-2\pm 2i\sqrt{3}}{2}$

$\boxed{x=-1\pm i\sqrt{3}}$

$x+2=0$

$\boxed{x=-2}$

$x^2-2x+4=0$

$x=\frac{2\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$

$x=\frac{2\pm\sqrt{-12}}{2}$

$x=\frac{2\pm 2i\sqrt{3}}{2}$

$\boxed{x=1\pm i\sqrt{3}}$

Example 3

Find all solutions of the equation by factoring.

$P(x) = 3x^5 + 24x^3 + 48x$

$0 = 3x(x^4 + 8x^2 + 16)$

S 8 P 16
4 4

$0 = 3x(x^2+4)(x^2+4)$

$3x=0$

$\boxed{x=0}$

$x^2+4=0$

$\sqrt{x^2}=\sqrt{-4}$

$\boxed{x=\pm 2i}$

Example 4

Find all solutions of the equation by factoring.

$0 = x^6 - 7x^3 - 8$

$s = -7 \quad p = -8$

$-8 \quad 1$

$1 \quad 1$

$0 = (x^3 - 8)(x^3 + 1)$

$0 = (x-2)(x^2+2x+4)(x+1)(x^2-x+1)$

$\boxed{x=2}$ $\qquad \boxed{x=-1}$

$x = \dfrac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$

$x = \dfrac{-2 \pm \sqrt{-12}}{2}$

$x = \dfrac{-2 \pm 2i\sqrt{3}}{2}$

$\boxed{x = -1 \pm i\sqrt{3}}$

$x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$x = \dfrac{1 \pm \sqrt{-3}}{2}$

$\boxed{x = \dfrac{1 \pm i\sqrt{3}}{2}}$

Example 5

Find all solutions of the equation by factoring.

$P(x) = (x^3 + 7x^2) + (18x + 18)$

~~$0 = x^2(x+7) + 18(x+1)$~~

$\dfrac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

-3	1	7	18	18
	↓	-3	-12	-18
	1	4	6	0

$0 = (x+3)(x^2+4x+6)$

$\boxed{x=-3}$

$x = \dfrac{-4 \pm \sqrt{(4)^2 - 4(1)(6)}}{2(1)}$

$x = \dfrac{-4 \pm \sqrt{-8}}{2}$

$x = \dfrac{-4 \pm 2i\sqrt{2}}{2}$

$\boxed{x = -2 \pm i\sqrt{2}}$

Example 6

Find all solutions of the equation by factoring.

$P(x) = x^4 + 2x^2 + 1$

s 2 p 1

\+ \+

$0 = (x^2+1)(x^2+1)$

$x^2+1=0$

$\sqrt{x^2} = \sqrt{-1}$

$\boxed{x = \pm i}$

The Conjugate Zeros Theorem *states that complex zeros come in conjugate pairs. In other words, if 2i is a zero, then -2i is also a zero. Or if 5 - 3i is a zero, then 5 + 3i is also a zero.*

Example 7

Find a polynomial $P(x)$ of degree 3 that has integer coefficients and zeros 4 and $2i$. & $-2i$

$P(x) = (x-4)\underbrace{(x-2i)(x+2i)}$

$P(x) = (x-4)(x^2+\cancel{2i}x-\cancel{2i}x-4\overset{(-1)}{\cancel{i^2}})$

$P(x) = (x-4)(x^2+4)$

$P(x) = x^3+4x-4x^2-16$

$\boxed{P(x) = x^3-4x^2+4x-16}$

Example 8

$x = 5+2i \qquad x = 5-2i$

Find a polynomial $P(x)$ of degree 2 that has integer coefficients and zeros $5 + 2i$ and $5 - 2i$.

$P(x) = [x-(5+2i)][x-(5-2i)]$

$P(x) = [x-5-2i][x-5+2i]$

$P(x) = x(x-5+2i)-5(x-5+2i)-2i(x-5+2i)$

$P(x) = x^2-5x+\cancel{2ix}-5x+25-\cancel{10i}-\cancel{2ix}+\cancel{10i}-4i^2$ (-1)

$P(x) = x^2-5x-5x+25+4$

$\boxed{P(x) = x^2-10x+29}$

Example 9

Find a polynomial $P(x)$ of degree 4 that has zeros i, $-i$, 2, and -2 and with $P(3) = 25$.

$P(x) = \underbrace{(x-i)(x+i)}\underbrace{(x-2)(x+2)}$

$P(x) = (x^2+\cancel{ix}-\cancel{ix}-i^2)(x^2+\cancel{2x}-\cancel{2x}-4)$

$P(x) = (x^2+1)(x^2-4)$

$P(x) = x^4-4x^2+x^2-4$

$P(x) = \frac{1}{2}(x^4-3x^2-4)$

$25 = a\,\underbrace{(3^4-3\cdot 3^2-4)}$

$\frac{25}{50} = \frac{a\cdot 50}{50}$

$\frac{1}{2} = a$

$\boxed{P(x) = \frac{1}{2}x^4-\frac{3}{2}x^2-2}$