3.4 Part 1 Real Zeros of Polynomials

Rational Root Theorem

If a polynomial has integer coefficients, then every rational zero has the form:

$$\boxed{\pm\frac{p}{q} = \pm\frac{\text{factor of the constant}}{\text{factor of the leading coefficient}}}$$

possible zeros

I. List All Possible Rational Zeros

EXAMPLES:

LC constant

1. $f(x) = 1x^3 + 2x^2 - 5x + 6$

p factors of constant term: $\pm 1, \pm 2, \pm 3, \pm 6$

q factors of leading coefficient: ± 1

$\frac{p}{q}$ possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

2. $f(x) = 2x^3 - x^2 + 5x + 6$

$p: \pm 1, \pm 2, \pm 3, \pm 6$

$q: \pm 1, \pm 2$

$\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

3. $f(x) = 6x^4 + 35x^3 + 35x^2 - 55x - 21$

$p: \pm 1, \pm 3, \pm 7, \pm 21$

$q: \pm 1, \pm 2, \pm 3, \pm 6$

$\frac{p}{q}: \pm 1, \pm 3, \pm 7, \pm 21, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{7}{2}, \pm \frac{21}{2},$

$\pm \frac{1}{3}, \pm \frac{7}{3}, \pm \frac{1}{6}, \pm \frac{7}{6}$

Fundamental Theorem of Algebra

A polynomial of degree n has exactly n roots (zeros) in the set of complex numbers.

Roots or zeros may be rational (integers or fractions), irrational (square roots), or imaginary (i).

II. Find ALL Zeros

STEPS:

1. List all possible roots.
2. Test each possibility until you find one zero.
3. Divide by the zero (using synthetic division) to get depressed polynomial.
4. Repeat steps 1 to 3 until the depressed polynomial is a quadratic.
5. Solve the quadratic by factoring, square roots, or the quadratic formula to get the last 2 zeros.

EXAMPLES:

4. $f(x) = x^3 - 6x^2 + 12x - 8$

p: ±1, ±2, ±4, ±8

q: ±1

$\frac{p}{q}$: ±1, ±2, ±4, ±8

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & \downarrow & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$(x-2)(x^2-4x+4)$

$(x-2)(x-2)(x-2)=0$

$\boxed{\text{zero}: 2}$

EXAMPLES:

5. $f(x) = 1x^3 + 3x^2 - 4$

p: ±1, ±2, ±4

q: ±1

$\frac{p}{q}$: ±1, ±2, ±4

$$\begin{array}{r|rrrr} 1 & 1 & 3 & 0 & -4 \\ & \downarrow & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$(x-1)(x^2+4x+4)$

$(x-1)(x+2)(x+2)=0$

$\boxed{\text{zeros: } 1, -2}$

EXAMPLES:

6. $f(x) = 2x^4 + 3x^3 - 11x^2 - 9x + 15$

p: ±1, ±3, ±5, ±15

q: ±1, ±2

$\frac{p}{q}$: $\pm1, \pm3, \pm5, \pm15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}$

$$\begin{array}{r|rrrrr} 5 & 2 & 3 & -11 & -9 & 15 \\ & \downarrow & 10 & 65 & & \\ \hline & 2 & 13 & 54 & & \end{array}$$

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & 3 & -11 & -9 & 15 \\ & \downarrow & 1 & 2 & -\frac{9}{2} & -\frac{27}{4} \\ \hline & 2 & 4 & -9 & -\frac{27}{2} & \end{array}$$

$$\begin{array}{r|rrrrr} -3 & 2 & 3 & -11 & -9 & 15 \\ & \downarrow & -6 & 9 & 6 & 9 \\ \hline & 2 & -3 & -2 & -3 & \end{array}$$

$$\begin{array}{r|rrrrr} 1 & 2 & 3 & -11 & -9 & 15 \\ & \downarrow & 2 & 5 & -6 & -15 \\ \hline & 2 & 5 & -6 & -15 & 0 \end{array}$$

$(x-1)(2x^3+5x^2-6x-15)$

$x^2(2x+5)-3(2x+5)$

$(x-1)(2x+5)(x^2-3)=0$

$x=1 \qquad x=-\frac{5}{2} \qquad x^2-3=0$

$\sqrt{x^2}=\sqrt{3}$

$x=\pm\sqrt{3}$

$\boxed{\text{zeros: } 1, -\frac{5}{2}, \pm\sqrt{3}}$

EXAMPLES:

7. $f(x) = x^3 + 4x^2 + 3x - 2$

$p: \pm1, \pm2$

$q: \pm1$

$\frac{p}{q}: \pm1, \pm2$

$$\begin{array}{r|rrrr} -2 & 1 & 4 & 3 & -2 \\ & \downarrow & -2 & -4 & 2 \\ \hline & 1 & 2 & -1 & 0 \end{array}$$

$(x+2)(x^2+2x-1)$

$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)}$

$x = \frac{-2 \pm \sqrt{8}}{2}$

$x = \frac{-2 \pm 2\sqrt{2}}{2}$

$\boxed{x = -1 \pm \sqrt{2}, -2}$

EXAMPLES:

8. $f(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$

$p: \pm1, \pm2, \pm5, \pm10$

$q: \pm1$

$\frac{p}{q}: \pm1, \pm2, \pm5, \pm10$

$$\begin{array}{r|rrrrr} 5 & 1 & -5 & -5 & 23 & 10 \\ & \downarrow & 25 & 100 & & \\ \hline & 5 & 20 & 95 & & 11 \end{array}$$

Wingard goofed

$$\begin{array}{r|rrrrr} -2 & 1 & -5 & -5 & 23 & 10 \\ & \downarrow & -2 & 14 & -18 & -10 \\ \hline & 1 & -7 & 9 & 5 & 0 \end{array}$$

$(x+2)(x^3-7x^2+9x+5)$

$$\begin{array}{r|rrrr} 5 & 1 & -7 & 9 & 5 \\ & \downarrow & 5 & -10 & -5 \\ \hline & 1 & -2 & -1 & 0 \end{array}$$

$(x+2)(x-5)(x^2-2x-1)$

$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$ 8

$x = \frac{2 \pm 2\sqrt{2}}{2}$

$\boxed{x = 1 \pm \sqrt{2}, -2, 5}$

$$\begin{array}{r|rrrrr} -5 & 1 & -5 & -5 & 23 & 10 \\ & \downarrow & -5 & 50 & & \\ \hline & 1 & -10 & 45 & & \end{array}$$

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 9 & 5 \\ & \downarrow & 1 & -6 & 3 \\ \hline & 1 & -6 & 3 & 8 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 9 & 5 \\ & \downarrow & 2 & -10 & -2 \\ \hline & 1 & -5 & -1 & 3 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & -7 & 9 & 5 \\ & \downarrow & -1 & 8 & -17 \\ \hline & 1 & -8 & 17 & -12 \end{array}$$