5.6 Operations with Complex Numbers

Square Root of Negative Numbers

The square root of a negative real number has TWO imaginary roots: one positive, one negative.

$$\sqrt{-r} = \sqrt{-1} \cdot \sqrt{r} \quad \left(\text{where } \sqrt{-1} = i\right) = i\sqrt{r}$$

and

$$i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = -1$$

Examples: Simplify.

1. $\sqrt{-81} = 9i$

2. $\sqrt{-120} = 2i\sqrt{30}$

$2 \,|\, 120$
$2 \,|\, 60$
$2 \,|\, 30$
$3 \,|\, 15$
5

3. $\sqrt{-48} = 4i\sqrt{3}$

$2 \,|\, 48$
$2 \,|\, 24$
$2 \,|\, 12$
$2 \,|\, 6$
3

4. $\sqrt{-256} = 16i$

Examples: Solve using the indicated method.

5. $x^2 + 6x + 10 = 0$

quadratic formula

$a=1 \quad b=6 \quad c=10$

$x = \dfrac{-6 \pm \sqrt{(6)^2 - 4(1)(10)}}{2(1)}$

$x = \dfrac{-6 \pm \sqrt{-4}}{2}$

$x = \dfrac{-6 \pm 2i}{2}$

$x = \dfrac{-6}{2} \pm \dfrac{2i}{2}$

$\boxed{x = -3 \pm i}$

6. $-3x^2 - 10 = 44$

taking square roots

$-3x^2 - \cancel{10} = 44$

$\quad\ \ \cancel{+10} \quad +10$

$\dfrac{-3x^2}{-3} = \dfrac{54}{-3}$

$\sqrt{x^2} = \sqrt{-18}$

$2 \overline{)18}$ 9

$\boxed{x = \pm 3i\sqrt{2}}$

Examples: Solve using the indicated method.

7. $2(x - 1)^2 + 12 = 0$

taking square roots

$2(x-1)^2 + 12 = 0$

$\qquad\qquad -12 \quad -12$

$\dfrac{2(x-1)^2}{2} = \dfrac{-12}{2}$

$\sqrt{(x-1)^2} = \sqrt{-6}$

$x - 1 = \pm i\sqrt{6}$

$\quad +1 \quad +1$

$\boxed{x = 1 \pm i\sqrt{6}}$

8. $x^2 + 4x + 20 = 0$

completing the square

$x^2 + 4x + 20 = 0$

$\qquad\qquad -20 \quad -20$

$x^2 + 4x + 4 = -20 + 4$

$\frac{1}{2}(4) = 2$

$(2)^2 = 4$

$\sqrt{(x+2)^2} = \sqrt{-16}$

$x + 2 = \pm 4i$

$\quad -2 \quad -2$

$\boxed{x = -2 \pm 4i}$

Examples: Solve using any method.

9. $-\frac{1}{3}(x - 7)^2 + 5 = 23$

$-5 \quad -5$

$-\frac{3}{1} \cdot -\frac{1}{3}(x-7)^2 = 18 \cdot -\frac{3}{1}$

$\sqrt{(x-7)^2} = \sqrt{-54}$

$x - 7 = \pm 3i\sqrt{6}$

$+7 \quad +7$

$\boxed{x = 7 \pm 3i\sqrt{6}}$

$2|54$, $3|27$, $3|9$, 3

10. $3x^2 - 5x = -4$

$+4 \quad +4$

$3x^2 - 5x + 4 = 0$

$a=3 \quad b=-5 \quad c=4$

$x = \frac{5 \pm \sqrt{(-5)^2 - 4(3)(4)}}{2(3)}$

$x = \frac{5 \pm \sqrt{-23}}{6}$

$\boxed{x = \frac{5 \pm i\sqrt{23}}{6}}$

The standard form of a complex number is $a + bi$

real part: a ; imaginary part: bi

Every number can be written as a complex number.

$9 + 2i \longrightarrow$ imaginary number

$0 + 2i \longrightarrow$ pure imaginary number

$9 + 0i \longrightarrow$ real number

Adding and Subtracting Complex Numbers

Add or subtract: real part to real part
imaginary part to imaginary part

Examples: Simplify.

11. $(4 - i) + (3 + 2i)$

$7 + i$

12. $(7 - 5i) - (1 - 5i)$

$7 - \cancel{5i} - 1 + \cancel{5i}$

6

13. $6 - (-2 + 9i) + (-8 + 4i)$

$\cancel{6} + \cancel{2} - 9i - \cancel{8} + 4i$

$-5i$

14. $2i - (3 + i) + (2 - 3i)$

$2i - 3 - i + 2 - 3i$

$-1 - 2i$

$i^2 = -1$

Multiplying Complex Numbers

Examples: Simplify.

15. $5i(-2 + i)$

$-10i + 5i^2$

$-10i + 5(-1)$

$-10i - 5 \longrightarrow -5 - 10i$

16. $(-1 + 2i)(7 - 4i)$

$-7 + 4i + 14i - 8i^2$

$-7 + 18i - 8(-1)$

$-7 + 18i + 8$

$1 + 18i$

17. $(6 + 3i)(6 - 3i)$

$36 - \cancel{18i} + \cancel{18i} - 9i^2$

$36 - 9(-1)$

$36 + 9$

45

18. $(2 + 5i)^2 \rightarrow (2 + 5i)(2 + 5i)$

$4 + 10i + 10i + 25i^2$

$4 + 20i + 25(-1)$

$4 + 20i - 25$

$-21 + 20i$

Dividing Complex Numbers

A. Pure Imaginary Denominator

Multiply numerator and denominator by i.

Examples: Simplify.

19. $\frac{(2+8i)}{i}\cdot\frac{i}{i} = \frac{2i+8i^2}{i^2} = \frac{2i+8(-1)}{-1} = \frac{2i-8}{-1}$

$-2i+8 = \boxed{8-2i}$

20. $\frac{(3+7i)}{2i}\cdot\frac{i}{i} = \frac{3i+7i^2}{2i^2} = \frac{3i+7(-1)}{2(-1)} = \frac{3i-7}{-2} = \boxed{\frac{-7+3i}{-2}}$

21. $\frac{(4-i)}{2i}\cdot\frac{i}{i} = \frac{4i-i^2}{2i^2} = \frac{4i-(-1)}{2(-1)} = \frac{4i+1}{-2} = \boxed{\frac{1+4i}{-2}}$

B. Imaginary Denominator a + bi

To simplify a fraction with an imaginary number, a + bi, in the denominator, you must multiply by the conjugate of the denominator .

Example: The conjugate of 1 - 3i is 1 + 3i.

The conjugate of 6 + 5i is 6 - 5i.

Examples: Simplify.

22. $\frac{2}{(7-8i)}\cdot\frac{7+8i}{(7+8i)} = \frac{2(7+8i)}{49+\cancel{56i}-\cancel{56i}-64i^2} = \frac{2(7+8i)}{49-\cancel{64(-1)}}$ $+64$ $= \frac{2(7+8i)}{113}$

$\boxed{\frac{14+16i}{113}}$

Examples: Simplify.

23. $\frac{-3}{(2+i)} \cdot \frac{2-i}{(2-i)}$

$\frac{-3(2-i)}{4-\cancel{2i}+\cancel{2i}-i^2}$

$\frac{-3(2-i)}{4-(-1)}$

$\boxed{\frac{-3(2-i)}{5} \text{ or } \frac{-6+3i}{5}}$

24. $\frac{(3+11i)}{(-1-2i)} \cdot \frac{(-1+2i)}{(-1+2i)}$

$\frac{-3\boxed{+6i-11i}+22i^2}{1-\cancel{2i}+\cancel{2i}-4i^2}$

$\frac{-3-5i+\cancel{22(-1)}^{-22}}{1-\cancel{4(-1)}_{+4}}$

$\frac{-25-5i}{5}$

$\boxed{-5-i}$