2.7 One-to-One Functions and Their Inverses

I. Definition

The inverse of a set of ordered pairs (x, y) is the set of ordered pairs (y, x).

A function and its inverse "undo" each other.

Find the inverse of f.

$f = \{(0,1), (3,-2), (-7,4), (5,1)\}$

inverse $= \{(1,0), (-2,3), (4,-7), (1,5)\}$

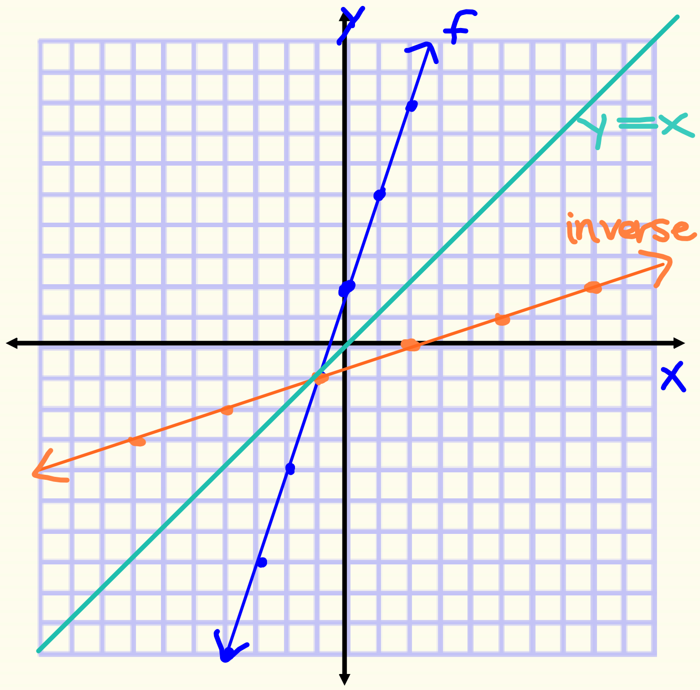
II. Graph a Function and Its Inverse

EXAMPLE:

Graph f (x) = 3x + 2.

$m=3$

$y\text{-int}=2$



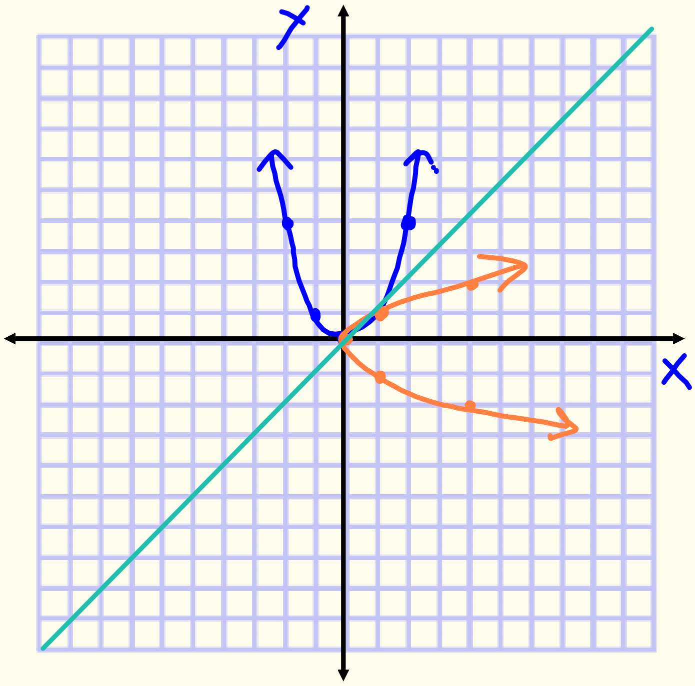
EXAMPLE:

Graph $f(x) = x^2$.

x	y
-2	4
-1	1
0	0
1	1
2	4

$\rightarrow$

x	y
4	-2
1	-1
0	0
1	1
4	2



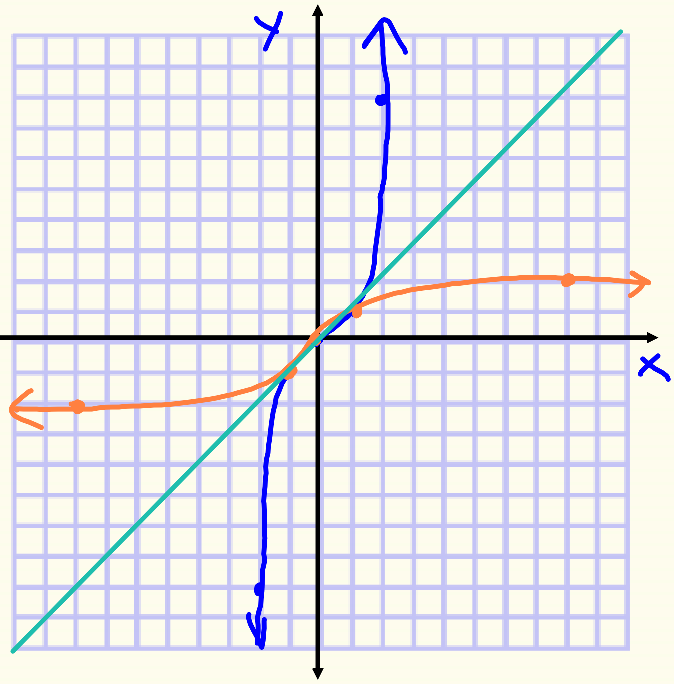
EXAMPLE:

Graph $f(x) = x^3$.

x	y
-2	-8
-1	-1
0	0
1	1
2	8

$\rightarrow$

x	y
-8	-2
-1	-1
0	0
1	1
8	2



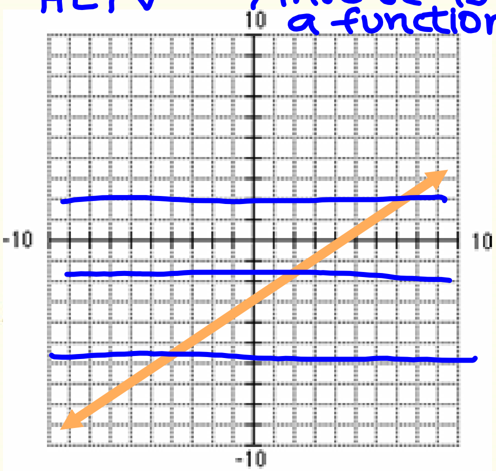
III. Determine if an Inverse is a Function

A. Graphically

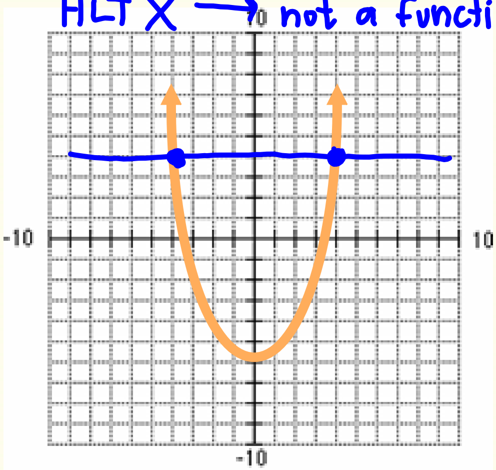
Horizontal Line Test (HLT):

If every horizontal line intersects the graph at exactly one point, then the inverse will be a function.

HLT ✓ → inverse is a function



HLT X → inverse is not a function



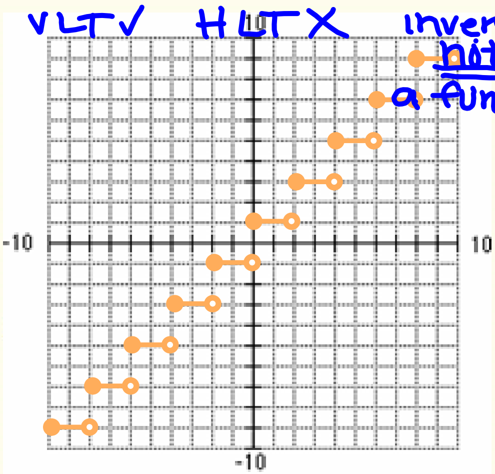
One-to-One Function:

If a function passes the horizontal line test, then it is a one-to-one function. This means that the function has an inverse function (no two x-values have the same y-value and no two y-values have the same x-value).

VLT ✓ & HLT ✓ → one-to-one function

State whether the following graphs have inverse functions.

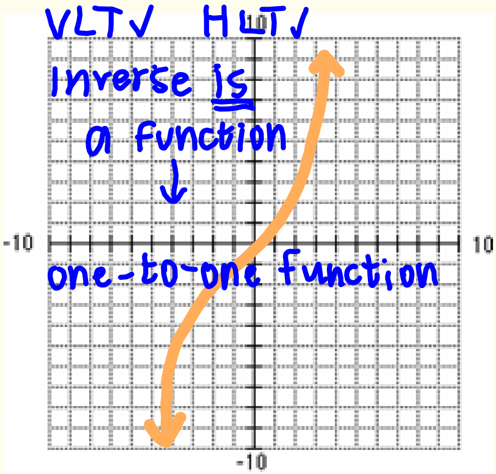
VLT ✓ HLT X inverse is not a function



VLT ✓ HLT ✓

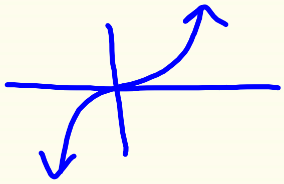
Inverse is a function ↓

one-to-one function



Are the following one-to-one functions?

$f(x) = x^3$

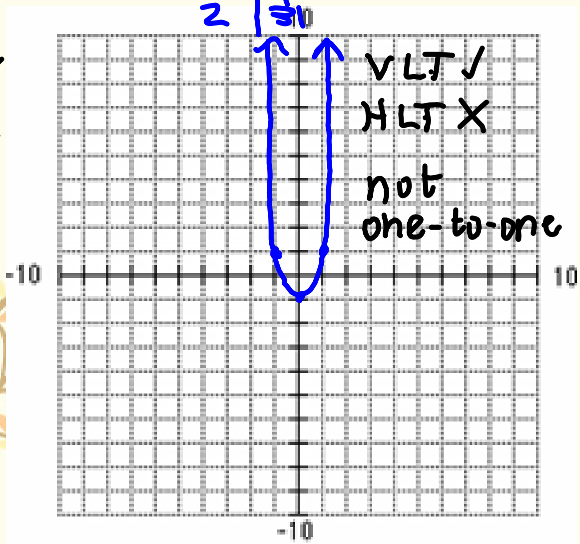


VLT✓ HLT✓

one-to-one

$g(x) = 2x^4 - 1$

x	y
-2	31
-1	1
0	-1
1	1
2	31

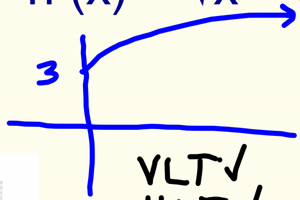


VLT✓

HLT X

not one-to-one

$h(x) = \sqrt{x} + 3$



VLT✓

HLT✓

one-to-one

B. Algebraically

f inverse

If f and f^{-1} are inverse functions then:

$$f \circ f^{-1} = x \text{ AND } f^{-1} \circ f = x$$

(f^{-1} undoes what f does, and vice versa)

EXAMPLE:

Show that f and g are inverse functions.

$f(x) = 4x - 3$ and $g(x) = \frac{1}{4}x + \frac{3}{4}$

$f \circ g$

$4(\frac{1}{4}x + \frac{3}{4}) - 3$

$x + \cancel{3} - \cancel{3}$

✓ x

$g \circ f$

$\frac{1}{4}(4x - 3) + \frac{3}{4}$

$x - \cancel{\frac{3}{4}} + \cancel{\frac{3}{4}}$

✓ x

f & g are inverses

EXAMPLE:

Show that f and g are inverse functions.

$f(x) = 2x - 5$ and $g(x) = \frac{x+5}{2}$

$f \circ g$

$\cancel{2}\left(\frac{x+5}{\cancel{2}}\right) - 5$

$x + \cancel{5} - \cancel{5}$

✓ x

$g \circ f$

$\frac{(2x - \cancel{5}) + \cancel{5}}{2}$

$\frac{\cancel{2}x}{\cancel{2}}$

✓ x

f & g are inverses

IV. Find the Inverse Function

To find the inverse function of an equation, swap the x and y variables.
Then solve for y. This will be the inverse $f^{-1}(x)$.

EXAMPLE:

State whether the function f has an inverse function. If f^{-1} exists, find a rule for $f^{-1}(x)$.

$f(x) = 4x - 7 \longrightarrow y = 4x - 7$

$x = 4y - 7$

$+7 \qquad +7$

$\frac{x+7}{4} = \frac{4y}{4}$

$\frac{x+7}{4} = y \longrightarrow \boxed{f^{-1}(x) = \frac{x+7}{4}}$

EXAMPLE:

State whether the function f has an inverse function. If f^{-1} exists, find a rule for $f^{-1}(x)$.

$f(x) = \sqrt[3]{x+4}$ HLT ✓ inv. is a function

$y = \sqrt[3]{x+4}$

$(x)^3 = \left(\sqrt[3]{y+4}\right)^3$

$x^3 = y + 4$

$-4 \qquad -4$

$x^3 - 4 = y$

$\boxed{f^{-1}(x) = x^3 - 4}$

EXAMPLE:

State whether the function f has an inverse function. If f^{-1} exists, find a rule for $f^{-1}(x)$.

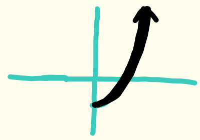
$f(x) = x^2 - 1$ HLT X inv. is not a function

EXAMPLE:

State whether the function f has an inverse function. If f^{-1} exists, find a rule for $f^{-1}(x)$.

y's of orig are x's of inv

$f(x) = x^2 - 1,\ \underline{x \geq 0}$ HLT ✓



x	y
0	-1
1	0
2	3
⋮	⋮

$y = x^2 - 1$

$x = y^2 - 1$

$+1 \qquad +1$

$\sqrt{x+1} = \sqrt{y^2}$

$\pm\sqrt{x+1} = y \longrightarrow f^{-1}(x) = \sqrt{x+1}\ ;\ x \geq -1$

EXAMPLE:

State whether the function f has an inverse function. If f^{-1} exists, find a rule for $f^{-1}(x)$.

$f(x) = 8x^3 + 1$ HLT ✓ inv. is a func.

$y = 8x^3 + 1$

$x = 8y^3 + 1$

$-1 \qquad -1$

$\frac{x-1}{8} = \frac{8y^3}{8}$

$\sqrt[3]{\frac{x-1}{8}} = \sqrt[3]{y^3}$

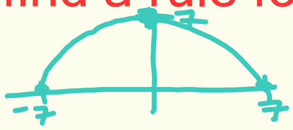
$\frac{\sqrt[3]{x-1}}{2} = y \longrightarrow f^{-1}(x) = \frac{\sqrt[3]{x-1}}{2}$

EXAMPLE:

$\sqrt{r^2 - x^2}$ semicircle

State whether the function f has an inverse function. If f^{-1} exists, find a rule for $f^{-1}(x)$.

$f(x) = \sqrt{49 - x^2}$ $r = 7$



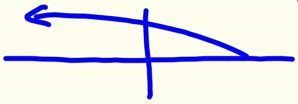
HLT X
inv. not a function

EXAMPLE:

$\sqrt{x}$

State whether the function f has an inverse function. If f^{-1} exists, find a rule for $f^{-1}(x)$.

$f(x) = \sqrt{8 - x} = \sqrt{-(x-8)}$



HLT ✓

$y = \sqrt{8-x}$

$(x)^2 = (\sqrt{8-y})^2$

$x^2 = 8 - y$

$x^2 - 8 = -y$

$-x^2 + 8 = y \longrightarrow f^{-1}(x) = -x^2 + 8$