2.4 Average Rate of Change

If you drive 120 miles in 2 hours, then your average speed, or rate of travel, is 60 mph. Suppose you take a car trip and record the distance every few minutes. The distance s you have traveled is a function of the time t:

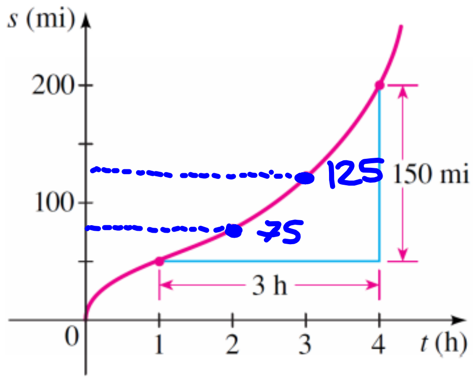
s(t) = total distance traveled at time t

To find your average speed between any two points on the trip, divide the distance traveled by the time elapsed.

Example 1

a) Calculate your average speed between 1:00 and 4:00.

$$\frac{150\text{miles}}{3\text{ hr}}$$
$$\downarrow$$
$$\boxed{50\text{ mph}}$$



b) Calculate your average speed between 2:00 and 3:00.

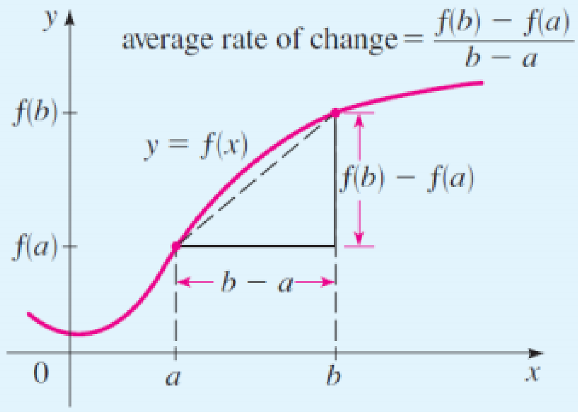
$$\frac{125-75}{3{:}00-2{:}00} = \frac{50}{1\text{hr}} = \boxed{50\text{ mph}}$$

Average Rate of Change

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change is the slope of the **secant line** between $x = a$ and $x = b$ on the graph of f, that is, the line that passes through $(a, f(a))$ and $(b, f(b))$.



Algebra Concept

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Precalculus Concept

$$\text{arc} = \frac{f(b) - f(a)}{b - a}$$

$f(a)$ & $f(b)$ → y-values

a & b → x-values

Example 2

For the function $f(x) = x^2 + 4$, find the average rate of change of the function between the following points:

a) $x = 2$ and $x = 6$

$f(2) = 2^2 + 4$
$= 8$

$f(6) = 6^2 + 4$
$= 40$

$$arc = \frac{f(6) - f(2)}{6 - 2} = \frac{40 - 8}{4} = \frac{32}{4}$$

$$\boxed{arc = 8}$$

b) $x = 5$ and $x = 10$

$f(5) = 5^2 + 4$
$= 29$

$f(10) = 10^2 + 4$
$= 104$

$$arc = \frac{f(10) - f(5)}{10 - 5} = \frac{104 - 29}{5} = \frac{75}{5}$$

$$\boxed{arc = 15}$$

Example 2

For the function $f(x) = x^2 + 4$, find the average rate of change of the function between the following points:

c) $x = a$ and $x = a + h$

$f(a) = a^2 + 4$

$f(a+h) = (a+h)^2 + 4$
$= a^2 + 2ah + h^2 + 4$

$$arc = \frac{f(a+h) - f(a)}{(\cancel{a}+h) - (\cancel{a})}$$

$$= \frac{(a^2 + 2ah + h^2 + 4) - (a^2 + 4)}{h}$$

$$= \frac{\cancel{a^2} + 2ah + h^2 + \cancel{4} - \cancel{a^2} - \cancel{4}}{h}$$

$$= \frac{\cancel{h}(2a + h)}{\cancel{h}}$$

$$\boxed{arc = 2a + h}$$

The average rate of change in example c is known as a difference quotient. In calculus, we use difference quotients to calculate instantaneous rates of change.

Example 3

If an object is dropped from a height of 3000 feet, its distance above the ground (in feet) after t seconds is given by $h(t) = 3000 - 16t^2$. Find the object's average speed for the following times:

a) between 1 and 2 seconds

$h(1) = 3000 - 16(1)^2$
$= 2984$

$h(2) = 3000 - 16(2)^2$
$= 2936$

$arc = \frac{h(2) - h(1)}{2-1}$

$= \frac{2936 - 2984}{1}$

$\boxed{arc = -48 \text{ ft/sec}}$

b) between 4 and 5 seconds

$h(4) = 3000 - 16(4)^2$
$= 2744$

$h(5) = 3000 - 16(5)^2$
$= 2600$

$arc = \frac{h(5) - h(4)}{5-4}$

$= \frac{2600 - 2744}{1}$

$\boxed{arc = -144 \text{ ft/sec}}$

time	temp (F)
8:00 am	38
9:00 am	40
10:00 am	44
11:00 am	50
12:00 pm	56
1:00 pm	62
2:00 pm	66
3:00 pm	67
4:00 pm	64
5:00 pm	58
6:00 pm	55
7:00 pm	51

Example 4

The table gives the outdoor temperature observed by a science student on a spring day. Find the average rate of change in temperature between the following times:

a) 8:00 am and 9:00 am
b) 1:00 pm and 3:00 pm
c) 4:00 pm and 7:00 pm

a) $arc = \frac{T(9) - T(8)}{9-8} = \frac{40-38}{1} = \boxed{2°F/hr}$

b) $arc = \frac{T(3) - T(1)}{3-1} = \frac{67-62}{2} = \frac{5}{2} = \boxed{2.5°F/hr}$

c) $arc = \frac{T(7) - T(4)}{7-4} = \frac{51-64}{3} = \boxed{\frac{-13}{3}°F/hr}$

Example 5

Let f(x) = 3x - 5. Find the average rate of change of f(x) between the following points:

a) x = 0 and x = 1

$f(0) = 3(0) - 5$
$= -5$

$f(1) = 3(1) - 5$
$= -2$

$$arc = \frac{f(1) - f(0)}{1 - 0}$$
$$= \frac{-2 - (-5)}{1}$$
$$\boxed{arc = 3}$$

b) x = 3 and x = 7

$f(3) = 3(3) - 5$
$= 4$

$f(7) = 3(7) - 5$
$= 16$

$$arc = \frac{f(7) - f(3)}{7 - 3}$$
$$= \frac{16 - 4}{4}$$
$$= \frac{12}{4}$$
$$\boxed{arc = 3}$$

Example 5

Let f(x) = 3x - 5. Find the average rate of change of f(x) between the following points:

c) x = a and x =a + h

$f(a) = 3(a) - 5$
$= 3a - 5$

$f(a+h) = 3(a+h) - 5$
$= 3a + 3h - 5$

$$arc = \frac{f(a+h) - f(a)}{(\cancel{a}+h) - (\cancel{a})}$$
$$= \frac{(\cancel{3a} + 3h - \cancel{5}) + {}^{-}(\cancel{3a} + \cancel{5})}{h}$$
$$= \frac{3h}{h}$$
$$\boxed{arc = 3}$$

What conclusion can you draw from your answers?

For a linear function, f(x) = mx + b, the average rate of change between any two points is the slope of the line. Why is this true?