6.6 The Law of Cosines

Use the Law of Cosines to solve triangles when given SSS or SAS.

Law of Cosines

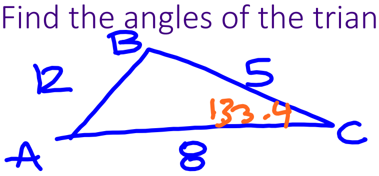
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Use the Law of Sines to find the smallest angle since the Law of Sines always gives the acute angle measure.

Use the Law of Cosines to find the largest angle since the Law of Cosines will give either the acute or obtuse angle.

Example 1

The sides of a triangle are a = 5, b = 8, and c = 12.
Find the angles of the triangle.

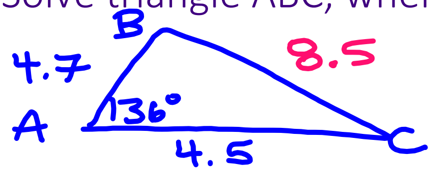


$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$12^2 = 5^2 + 8^2 - 2\cdot 5\cdot 8\cos C$$
$$144 = \cancel{25 + 64}^{\,89} - 80\cos C$$
$$-89 \qquad -89$$
$$\frac{55}{-80} = \frac{-80\cos C}{-80}$$
$$\cos^{-1}\left(-\frac{55}{80}\right) = \cancel{\cos^{-1}}\left(\cancel{\cos}\ C\right)$$
$$\boxed{m\angle C \approx 133.4^\circ}$$

$$\frac{\sin 133.4^\circ}{12} = \frac{\sin A}{5}$$
$$\sin^{-1}\left(\frac{5\sin 133.4^\circ}{12}\right) = \sin^{-1}\left(\sin A\right)$$
$$\boxed{\begin{array}{l} m\angle A \approx 17.6^\circ \\ m\angle B \approx 29.0^\circ \end{array}}$$

Example 2

Solve triangle ABC, where A = 136°, b = 4.5, and c = 4.7.



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\sqrt{a^2} = \sqrt{(4.5)^2 + (4.7)^2 - 2(4.5)(4.7)\cos 136°}$$

$$\boxed{a \approx 8.5}$$

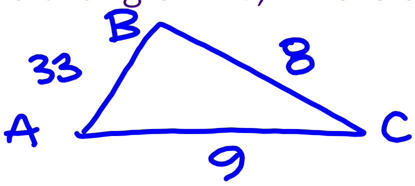
$$\frac{\sin B}{4.5} = \frac{\sin 136°}{8.5}$$

$$\sin^{-1}(\sin B) = \sin^{-1}\left(\frac{4.5\sin 136°}{8.5}\right)$$

$$\boxed{\begin{array}{c} m\angle B \approx 21.6° \\ m\angle C \approx 22.4° \end{array}}$$

Example 3

Solve triangle ABC, where a = 8 , b = 9, and c = 33.



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$33^2 = 8^2 + 9^2 - 2(8)(9)\cos C$$

$$1089 = \overset{145}{\cancel{64+81}} - 144\cos C$$

$$\underline{-145 \qquad\qquad -145}$$

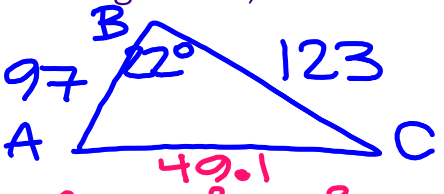
$$944 = -144\cos C$$

$$\cos^{-1}\left(\frac{944}{-144}\right) = \cos^{-1}(\cos C)$$

$$\boxed{\text{no triangle}}$$

Example 4

Solve triangle ABC, where B = 22°, a = 123, and c = 97.



$b^2 = a^2 + c^2 - 2ac\cos B$

$\sqrt{b^2} = \sqrt{(123)^2 + (97)^2 - 2(123)(97)\cos 22°}$

$\boxed{b \approx 49.1}$

$\dfrac{\sin 22°}{49.1} = \dfrac{\sin C}{97}$

$\sin^{-1}\left(\dfrac{97\sin 22°}{49.1}\right) = \sin^{-1}(\sin C)$

$\boxed{\begin{array}{l} m\angle C \approx 47.7° \\ m\angle A \approx 110.3° \end{array}}$

Example 5

A pilot sets out from an airport and heads in the direction N 20° E, flying at 200 mi/hr. After one hour, he makes a course correction and heads in the direction N 40° E. Half an hour after that, engine trouble forces him to make an emergency landing.

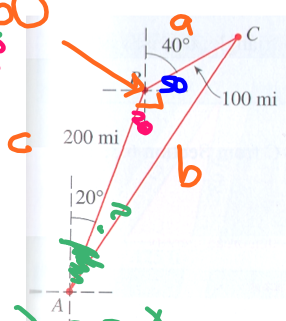
a) Find the distance between the airport and his final landing point. AC

b) Find the bearing from the airport to his final landing point.

a) $b^2 = a^2 + c^2 - 2ac\cos B$ 160°

$\sqrt{b^2} = \sqrt{(100)^2 + (200)^2 - 2(100)(200)\cos 160°}$

$\boxed{b \approx 295.95 \text{ miles}}$



b) Find $\angle BAC$

$a^2 = b^2 + c^2 - 2bc\cos A$

$100^2 = 295.95^2 + 200^2 - 2(295.95)(200)\cos A$

$10000 = 87586.4025 + 40000 - 118380\cos A$

$-117586.4025 = -118380\cos A$

$\cos^{-1}\left(\dfrac{117586.4025}{118380}\right) = \cos^{-1}(\cos A)$

$m\angle A \approx 6.6°$

$\boxed{N\ 26.6°\ E}$

Area of a Triangle using Heron's Formula

$$\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$$

where s = semiperimeter
a, b, & c = the lengths of the sides of the triangle

semiperimeter = $\frac{1}{2}(a + b + c)$

Example 6
A businessman wishes to buy a triangular lot in a busy downtown location. The lot frontages on the three adjacent streets are 125, 280, and 315 feet. Find the area of the lot.

$s = \frac{1}{2}(125 + 280 + 315)$

$s = 360$

$A = \sqrt{360(360-125)(360-280)(360-315)}$

$$\boxed{A \approx 17451.6 \text{ ft}^2}$$