4.6 Modeling with Exponential and Logarithmic Functions

Exponential functions - population growth, radioactive decay, heat diffusion, and others

Logarithmic functions - loudness of sounds, intensity of earthquakes, and other phenomena

Population

Biologists have observed that the population of a species doubles its size in a fixed period of time. For example, under ideal conditions a certain population of bacteria doubles in size every 3 hours. If the culture started with 1000 bacteria, then after 3 hours there will be 2000 bacteria, after another 3 hours there will be 4000 bacteria, and so on. If we let $n = n(t)$ be the number of bacteria after t hours, then

$$n(0) = 1000$$
$$n(3) = 2000$$
$$n(6) = 4000$$
$$n(9) = 8000$$

From this pattern it appears that the number of bacteria after t hours is modeled by the function

$$n(t) = 1000(2)^{\frac{t}{3}}$$

Example 1: A specific type of animal was introduced in a rural region 26 years ago. Biologists observe that the population quadruples every 8 years, and now the population is 11,425.

base = 4 $\frac{t}{8}$ $n(26)$

a) What was the initial population?

$$\frac{11{,}425}{4^{26/8}} = \frac{a\,\cancel{(4)^{26/8}}}{\cancel{4^{26/8}}}$$

$a \approx 126$ animals

b) Find a function that models the number of bacteria after t hours. $n(t) = 126(4)^{\frac{t}{8}}$

c) Estimate the population 15 years from now.

$n(41) = 126(4)^{41/8}$

$n(41) \approx 153{,}436$ animals

$n(15) = 11425(4)^{15/8}$

$n(15) \approx 153{,}716$ animals

Exponential Growth Model

$$n(t) = n_0 e^{rt}$$

where $n(t)$ = population at time t

n_0 = initial size of population

r = relative rate of growth (expressed as a proportion of the population)

t = time

Example 2: The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the relative growth is 40% per hour.

(0, 500)

$r = .40$

a) Find a function that models the number bacteria after t hours.

$n(t) = 500e^{.4t}$

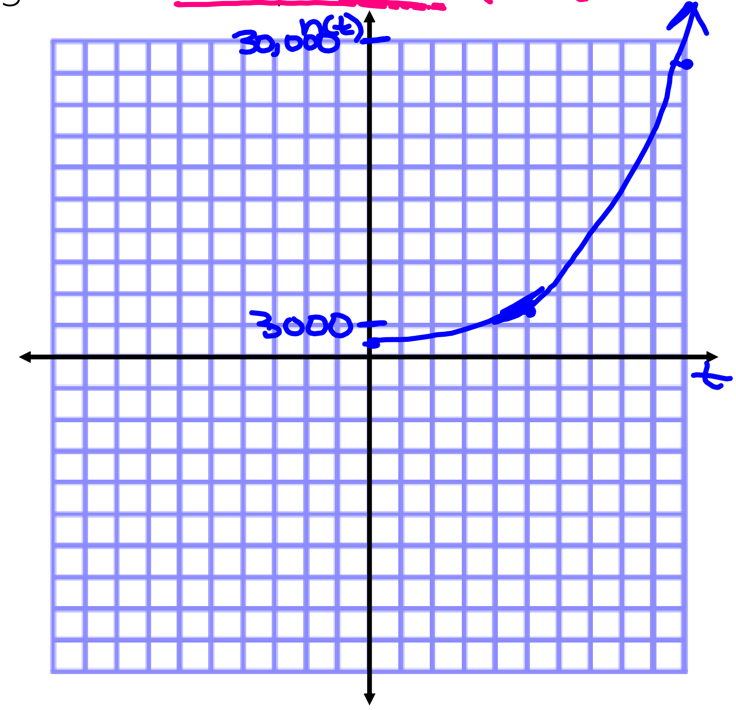
b) What is the estimated count after 10 hours?

$n(10) = 500e^{(.4)(10)}$

$n(10) \approx 27,299$ people

c) Sketch the graph of the function n(t).

$n(5) \approx 3695$



Example 3: In 2000 the population of the world was 6.1 billion and the relative rate of growth was 1.4% per year. It is claimed that a rate of 1.0% per year would make a significant difference in the total population in just a few decades. Test this claim by estimating the population of the world in the year 2050 using a relative rate of growth of

$t = 50$

a) 1.4% per year $r = .014$

$n(50) = 6.1e^{(.014)(50)}$

$n(50) \approx 12.284$ billion

b) 1.0% per year $r = .01$

$n(50) = 6.1e^{(.01)(50)}$

$n(50) \approx 10.057$ billion

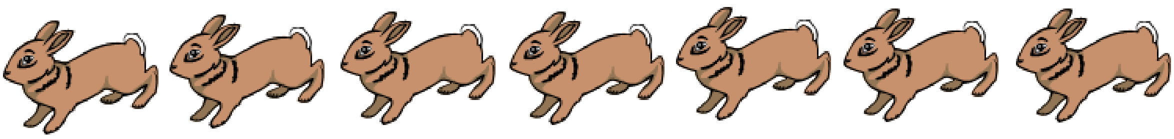
Example 4: The population of the world in 2000 was 6.1 billion, and the estimated relative growth rate was 1.4% per year. If the population continues to grow at this rate, when will it reach 122 billion? $n(t)$ Find t.

$$\frac{122}{6.1} = \frac{6.1e^{.014t}}{6.1}$$

$$\ln 20 = \ln e^{.014t}$$

$$\frac{\ln 20}{.014} = \frac{.014t}{.014}$$

$214 \approx t$
years



Example 5: A certain breed of rabbit was introduced onto a small island about 8 years ago. The current rabbit population on the island is estimated to be 4100, with a relative growth rate of 55% per year.

$n(8)$ $r=.55$

$$n(t)=n_0 e^{rt}$$

a) What was the initial size of the rabbit population?

$$\frac{4100}{e^{(.55)(8)}} = \frac{n_0 \cancel{e^{(.55)(8)}}}{\cancel{e^{(.55)(8)}}}$$

$n_0 \approx 50$ rabbits

b) Estimate the population 12 years from now.

$$n(20)=50e^{(.55)(20)}$$

$n(20) \approx 2,993,707$
rabbits

$$n(12)=4100e^{(.55)(12)}$$

$n(12) \approx 3,013,980$
rabbits

Example 6: A culture starts with 10,000 bacteria, and the number doubles every 40 minutes.

n_0

a) Find a function that models the number of bacteria at time t.
b) Find the number of bacteria after one hour. $n=60$
c) After how many minutes will there be 50,000 bacteria? $n(t)$

a) $n(t) = 10{,}000(2)^{\frac{t}{40}}$

b) $n(60) = 10{,}000(2)^{\frac{60}{40}}$

$n(60) \approx 28{,}284$ bacteria

c) $\frac{50{,}000}{10{,}000} = \frac{10{,}000(2)^{\frac{t}{40}}}{10{,}000}$

$\log 5 = \log(2)^{\frac{t}{40}}$

$40 \cdot \frac{\log 5}{\log 2} = \frac{\frac{t}{40}\log 2}{\log 2} \cdot 40$

$\boxed{93 \approx t \text{ min}}$

In 1935 the American geologist Charles Richter defined the magnitude M of an earthquake to be $M = \log \frac{I}{S}$

where I is the intensity of the earthquake

and S is the intensity of a "standard" earthquake.

Example 7: The 1906 earthquake in San Francisco had an estimated magnitude of 8.3 on the Richter scale. In the same year a powerful earthquake occured on the Colombia-Ecuador border that was 4 times as intense. What was the magnitude of the Colombia-Ecuador earthquake on the Richter scale?

$M = \log \frac{4I}{S}$

$M = \log 4 + \boxed{\log \frac{I}{S}}$

$M = \log 4 + 8.3$

$\boxed{M \approx 8.9}$