7.5 Part 1 Real Zeros of Polynomials

Rational Root Theorem

If a polynomial has integer coefficients, then every rational zero has the form:

possible zeros

$$\boxed{\pm\frac{p}{q} = \pm\frac{\overset{p}{\text{factor of the constant}}}{\underset{q}{\text{factor of the leading coefficient}}}}$$

I. List All Possible Rational Zeros

EXAMPLES:

1. $f(x) = \overset{q}{1}x^3 + 2x^2 - 5x + \overset{p}{6}$

p factors of constant term: ±1, ±2, ±3, ±6

q factors of leading coefficient: ±1

$\frac{p}{q}$ possible rational zeros: ±1, ±2, ±3, ±6

2. $f(x) = 2x^3 - x^2 + 5x + 6$

p: $\pm 1, \pm 2, \pm 3, \pm 6$

q: $\pm 1, \pm 2$

$\frac{p}{q}$: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

3. $f(x) = 6x^4 + 35x^3 + 35x^2 - 55x - 21$

p: $\pm 1, \pm 3, \pm 7, \pm 21$

q: $\pm 1, \pm 2, \pm 3, \pm 6$

$\frac{p}{q}$: $\pm 1, \pm 3, \pm 7, \pm 21, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{7}{2}, \pm \frac{21}{2},$
$\pm \frac{1}{3}, \pm \frac{7}{3}, \pm \frac{1}{6}, \pm \frac{7}{6}$

Fundamental Theorem of Algebra

A polynomial of degree n has exactly n roots (zeros) in the set of complex numbers.

Roots or zeros may be rational (integers or fractions), irrational (square roots), or imaginary (i).

II. Find ALL Zeros

STEPS:

1. List all possible roots. p/q
2. Test each possibility until you find one zero.
3. Divide by the zero (using synthetic division) to get depressed polynomial.
4. Repeat steps 1 to 3 until the depressed polynomial is a quadratic.
5. Solve the quadratic by factoring, square roots, or the quadratic formula to get the last 2 zeros.

EXAMPLES: Find all the roots.

4. $f(x) = x^3 - 5x^2 + 3x + 9$

$p: \pm 1, \pm 3, \pm 9$

$q: \pm 1$

possible zeros $\frac{p}{q}: \pm 1, \pm 3, \pm 9$

$$\begin{array}{r|rrrr} 3 & 1 & -5 & 3 & 9 \\ & \downarrow & 3 & -6 & -9 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

$(x-3)(x^2-2x-3)$

$(x-3)(x-3)(x+1)$

$x-3=0 \quad x-3=0 \quad x+1=0$

$\boxed{x=3,-1}$

EXAMPLES: Find all the roots.

5. $f(x) = x^3 + 3x^2 - 4$

$p: \pm 1, \pm 2, \pm 4$

$q: \pm 1$

$\frac{p}{q}: \pm 1, \pm 2, \pm 4$

$$\begin{array}{r|rrrr} 2 & 1 & 3 & 0 & -4 \\ & \downarrow & 2 & 10 & 20 \\ \hline & 1 & 5 & 10 & 16 \end{array}$$

$$\begin{array}{r|rrrr} 4 & 1 & 3 & 0 & -4 \\ & \downarrow & 4 & 28 & 112 \\ \hline & 1 & 7 & 28 & \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 3 & 0 & -4 \\ & \downarrow & -2 & -2 & 4 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$(x+2)(x^2+x-2)$

$(x+2)(x+2)(x-1)$

$\boxed{x = -2, 1}$

TYPO ON YOUR PAPER!

EXAMPLES: Find all the roots.

6. $2x^4 + x^3 - 11x^2 - 5x + 5 = 0$

$p: \pm 1, \pm 5$

$q: \pm 1, \pm 2$

$\frac{p}{q}: \pm 1, \pm 5, \pm \frac{1}{2}, \pm \frac{5}{2}$

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & 1 & -11 & -5 & 5 \\ & \downarrow & 1 & 1 & -5 & -5 \\ \hline & 2 & 2 & -10 & -10 & 0 \end{array}$$

$(x-\frac{1}{2})(2x^3+2x^2-10x-10)$

$2(x-\frac{1}{2})((x^3+x^2)(-5x-5))$

$x^2(x+1) - 5(x+1)$

$(2x-1)(x+1)(x^2-5)$

$\boxed{x = \frac{1}{2}, -1, \pm\sqrt{5}}$

$x^2 - 5 = 0$

$+5 \quad +5$

$\sqrt{x^2} = \sqrt{5}$

$x = \pm\sqrt{5}$

EXAMPLES: Find all the roots.

7. $x^3 - 5x^2 + 11x - 10 = 0$

$p: \pm 1, \pm 2, \pm 5, \pm 10$

$q: \pm 1$

$\frac{p}{q}: \pm 1, \pm 2, \pm 5, \pm 10$

$$\begin{array}{r|rrrr} 2 & 1 & -5 & 11 & -10 \\ & \downarrow & 2 & -6 & 10 \\ \hline & 1 & -3 & 5 & 0 \end{array}$$

$(x-2)(x^2-3x+5)$

$\boxed{x = \frac{3 \pm i\sqrt{11}}{2}, 2}$

$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(5)}}{2(1)}$

$x = \frac{3 \pm \sqrt{-11}}{2}$

$x = \frac{3 \pm i\sqrt{11}}{2}$

EXAMPLES: Find all the roots.

8. $x^4 - 3x^3 - 5x^2 + 13x + 6 = 0$

$\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrrr} 3 & 1 & -3 & -5 & 13 & 6 \\ & \downarrow & 3 & 0 & -15 & -6 \\ \hline -2 & 1 & 0 & -5 & -2 & 0 \\ & \downarrow & -2 & 4 & 2 & \\ \hline & 1 & -2 & -1 & 0 & \end{array}$$

$(x-3)(x+2)(x^2-2x-1)$

$$\begin{array}{r|rrrrr} -2 & 1 & -3 & -5 & 13 & 6 \\ & \downarrow & -2 & 10 & -10 & -6 \\ \hline 3 & 1 & -5 & 5 & 3 & 0 \\ & \downarrow & 3 & -6 & -3 & \\ \hline & 1 & -2 & -1 & 0 & \end{array}$$

$(x+2)(x-3)(x^2-2x-1)$

$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = \frac{2}{2} \pm \frac{2\sqrt{2}}{2}$

$1 \pm \sqrt{2}$

$\boxed{\text{Zeros} = 3, -2, 1 \pm \sqrt{2}}$